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Monotone Approximation of Certain Classes of Functions†

JOHN A. ROULIER

Department of Mathematics, Michigan State University, East Lansing, Michigan 48823

1. INTRODUCTION

Let H_n $(n = 0, 1, \ldots)$ be the set of all algebraic polynomials of degree n or less. We define $H_{n,k}$, $n = 0, 1, \ldots$; $k = 0, 1, \ldots, n$, to be the set of all $P_n \in H_n$ such that $P_n^{(k)}(x) \geqslant 0$ on the interval $[0, 1]$. For $f \in C[0, 1]$, the degree of approximation to f by polynomials from H_n is

$$E_n(f) = \inf_{P_n \in H_n} \|f - P_n\|,$$

where $\|\cdot\|$ is the uniform norm.

Similarly, if $f^{(k-1)}$ exists and is increasing on $[0, 1]$,

$$E_{n,k}(f) = \inf_{P_n \in H_{n,k}} \|f - P_n\|$$

is the degree of approximation to f by polynomials from $H_{n,k}$.

The purpose of this paper is to find some upper bounds for $E_{n,k}(f)$.

O. Shisha [5] examined this problem and proved the following.

If $1 \leqslant k \leqslant p$ and

$$f^{(k)}(x) \geqslant 0, \qquad |f^{(p)}(x)| \leqslant M \qquad \text{for } 0 \leqslant x \leqslant 1,$$

then for every integer n $(\geqslant p)$,

$$E_{n,k}(f) \leqslant \frac{C}{n^{p-k}} \omega\left(f^{(p)}, \frac{1}{n}\right)$$

where C depends upon p and k. $\omega(g, h)$ is the modulus of continuity of the function g.

The estimates in this paper are in many cases better than Shisha's estimate.

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2. The Main Theorems

The first of our theorems does not assume that f is differentiable.

THEOREM 1. *Let $f \in C[0,1]$ have the property*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geqslant \rho > 0 \qquad \text{if } 0 \leqslant x_1 < x_2 \leqslant 1. \tag{1}$$

Then

$$E_{n,1}(f) \leqslant \omega\left(\frac{3E_n(f)}{\rho}\right) + E_n(f), \tag{2}$$

for n sufficiently large.

(ω is the modulus of continuity of f on $[0,1]$.)

Proof. Let $E_n(f) = E_n$ for brevity. Choose n sufficiently large to insure that $3E_n/\rho < 1$. Choose $Q_n \in H_n$ so that $\|f - Q_n\| = E_n$. We have then, using (1) and the definition of Q_n

$$\begin{aligned} Q_n(x_2) - Q_n(x_1) &\geqslant f(x_2) - f(x_1) - |f(x_1) - Q_n(x_1)| - |f(x_2) - Q_n(x_2)| \\ &\geqslant \rho(x_2 - x_1) - 2E_n \geqslant E_n > 0 \qquad \text{if } x_2 - x_1 \geqslant (3E_n)/\rho. \end{aligned} \tag{3}$$

Consider the polynomial of degree at most n,

$$P_n(x) = \frac{\rho}{3E_n} \int_{\alpha(x)}^{\beta(x)} Q_n(t)\, dt$$

where $\alpha(x) = (1 - 3E_n/\rho)x$ and $\beta(x) = \alpha(x) + 3E_n/\rho$. We have $0 \leqslant \alpha(x) < \beta(x) \leqslant 1$ if $0 \leqslant x \leqslant 1$. Using (3), we see that

$$\begin{aligned} P_n'(x) &= \frac{\rho}{3E_n}\left(1 - \frac{3E_n}{\rho}\right)[Q_n(\beta(x)) - Q_n(\alpha(x))] \\ &\geqslant \frac{\rho}{3E_n}\left(1 - \frac{3E_n}{\rho}\right)E_n > 0 \qquad \text{for } 0 \leqslant x \leqslant 1. \end{aligned} \tag{4}$$

If $\alpha(x) \leqslant t \leqslant \beta(x)$, then, using $\alpha(x) \leqslant x \leqslant \beta(x)$, we deduce

$$|f(x) - f(t)| \leqslant \omega\left(\frac{3E_n}{\rho}\right). \tag{5}$$

Using (5) and the definition of Q_n we have

$$\begin{aligned} |f(x) - P_n(x)| &= \frac{\rho}{3E_n}\left|\int_{\alpha(x)}^{\beta(x)} [f(x) - Q_n(t)]\, dt\right| \\ &\leqslant \frac{\rho}{3E_n}\int_{\alpha(x)}^{\beta(x)} |f(x) - f(t)|\, dt + \frac{\rho}{3E_n}\int_{\alpha(x)}^{\beta(x)} |f(t) - Q_n(t)|\, dt \\ &\leqslant \omega\left(\frac{3E_n}{\rho}\right) + E_n. \end{aligned} \tag{6}$$

This gives (2).

The following is a corollary to Theorem 1.

COROLLARY. *If f satisfies* (1) *and, in addition, belongs to a Lipschitz class* $\mathrm{Lip}_1\,\alpha$, $0<\alpha\leqslant 1$, *then*

$$E_{n,1}(f)\leqslant C(\rho^{-\alpha}n^{-\alpha^2}+n^{-\alpha}) \tag{7}$$

for n sufficiently large.

This follows from the estimate

$$E_n(f)\leqslant \text{const.}\, n^{-\alpha}.$$

In the following two theorems we use the degree of approximation of a function ϕ by its Bernstein polynomial $B_n(\phi)$ (see [2]), expressed in terms of the modulus of continuity of ϕ or of ϕ'.

THEOREM 2. *Suppose that* $f'\in C[0,1]$ *and* $f'(x)\geqslant\rho>0$ *on* $[0,1]$. *Then*

$$E_{n,1}(f)\leqslant\frac{5}{2n^{1/2}}E_{n-1}(f') \tag{8}$$

if n is sufficiently large.

Proof. Let $P_{n-1}\in H_{n-1}$ be the polynomial of best approximation to f' on $[0,1]$, $n=1,2,\ldots$. Choose n so large that $E_{n-1}(f')\leqslant\rho/2$. Then

$$Q_{n-1}(x)=P_{n-1}(x)-E_{n-1}(f')\geqslant 0\quad\text{on } [0,1],$$

and $\|f'-Q_{n-1}\|=2E_{n-1}(f')$.

Define

$$\phi(x)=f(x)-\int_0^x Q_{n-1}(t)\,dt.$$

Then $\phi'(x)=f'(x)-Q_{n-1}(x)\geqslant 0$ on $[0,1]$, and $\|\phi'\|=2E_{n-1}(f')$. Hence, $\phi\in\mathrm{Lip}_M 1$, where $M=2E_{n-1}(f')$. Then

$$\|\phi-B_n(\phi)\|\leqslant\frac{5}{4n^{1/2}}\cdot 2E_{n-1}(f'),$$

by [2], p. 20. That is, $\|f-P_n\|\leqslant(5/2n^{1/2})E_{n-1}(f')$ where

$$P_n(x)=B_n(\phi,x)+\int_0^x Q_{n-1}(t)\,dt.$$

But $B_n'(\phi,x)\geqslant 0$ on $[0,1]$, by [2], p. 23. Hence, we have

$$P_n'(x)=Q_{n-1}(x)+B_n'(\phi,x)\geqslant 0\quad\text{on } [0,1].$$

This gives (8).

THEOREM 3. *Let $k \geqslant 2$ be an integer, and suppose that $f^{(k)} \in C[0,1]$ and $f^{(k)}(x) \geqslant \rho > 0$ on $[0,1]$. Then, for n sufficiently large, we have*

$$E_{n,k}(f) \leqslant \frac{2}{n} E_{n-k}(f^{(k)}). \tag{9}$$

Proof. We first establish (9) for $k=2$. We assume that $f'' \in C[0,1]$ and $f''(x) \geqslant \rho > 0$ on $[0,1]$, and prove that for all sufficiently large n,

$$E_{n,2}(f) \leqslant \frac{3}{2n} E_{n-2}(f''). \tag{10}$$

Let Q_{n-2}^* be the polynomial of best approximation from H_{n-2} to f'' on $[0,1]$. Choose n large enough to insure that $E_{n-2}(f'') \leqslant \rho/2$.

Define $Q_{n-2}(x) = Q_{n-2}^*(x) - E_{n-2}(f')$. Then $Q_{n-2}(x) \geqslant 0$ on $[0,1]$ and $\|f'' - Q_{n-2}\| = 2E_{n-2}(f'')$.

Define

$$\phi_1(x) = f'(x) - \int_0^x Q_{n-2}(t)\, dt.$$

Then $\phi_1{}'(x) = f''(x) - Q_{n-2}(x) \geqslant 0$ on $[0,1]$, and $\|\phi_1{}'\| = 2E_{n-2}(f'')$.

Hence, $\phi_1 \in \mathrm{Lip}_M 1$, where $M = 2E_{n-2}(f'')$. Therefore, by [2], p. 21,

$$\|\phi - B_n(\phi)\| \leqslant \frac{3}{4n^{1/2}} \cdot 2E_{n-2}(f'') \frac{1}{n^{1/2}} = \frac{3}{2n} E_{n-2}(f''),$$

where

$$\phi(x) = \int_0^x \phi_1(t)\, dt = f(x) - R_n(x)$$

and R_n is a polynomial of degree at most n with the property $R_n''(x) \geqslant 0$.

Hence, $\|f - P_n\| \leqslant (3/2n) E_{n-2}(f'')$, where

$$P_n(x) = R_n(x) + B_n(\phi, x).$$

But, $P_n''(x) = R_n''(x) + B_n''(\phi, x) \geqslant 0$ on $[0,1]$. Hence, we have (10).

This shows that for n sufficiently large there is a polynomial P_{n-k+2} such that $P_{n-k+2}''(x) \geqslant 0$ on $[0,1]$ and

$$|f^{(k-2)}(x) - P_{n-k+2}(x)| \leqslant \frac{3}{2(n-k+2)} E_{n-k}(f^{(k)})$$

$$\leqslant \frac{2}{n} E_{n-k}(f^{(k)}).$$

It now follows by integrating $k-2$ times, that $\|f - Q_n\| \leqslant (2/n) E_{n-k}(f^{(k)})$ (for n sufficiently large), where Q_n is some polynomial of degree at most n. This completes the proof.

3. Remarks

We shall compare our results with those of Shisha and with estimates available from the theory of Bernstein polynomials. Let f satisfy (1) on $[0,1]$, and suppose that $f \in \mathrm{Lip}_M\,\alpha$ $(0 < \alpha \leqslant 1)$. Then from [2], p. 23, we see that the Bernstein polynomials $B_n(f,x)$ are increasing on $[0,1]$. Furthermore, by [2], p. 20, we have

$$|f(x) - B_n(f,x)| \leqslant \tfrac{5}{4}Mn^{-\alpha/2}, \qquad x \in [0,1].$$

Hence,

$$E_{n,1}(f) \leqslant \tfrac{5}{4}Mn^{-\alpha/2}. \tag{11}$$

The corollary to Theorem 1 gives

$$E_{n,1}(f) \leqslant Kn^{-\alpha^2} \tag{12}$$

for n sufficiently large. This is better than (11) if $\alpha > 1/2$.

Now suppose that f' exists and $0 < \rho \leqslant f'(x) \leqslant M$ for $0 \leqslant x \leqslant 1$. Then $f \in \mathrm{Lip}_M 1$ and f satisfies (1). Hence (2) gives

$$E_{n,1}(f) \leqslant \left(\frac{3M}{\rho} + 1\right) E_n(f) \tag{13}$$

for n sufficiently large. Using Jackson's theorem we obtain $E_{n,1}(f) \leqslant CMn^{-1}$, while Shisha's estimate gives only

$$E_{n,1}(f) \leqslant \text{const.}\, \omega\left(f', \frac{1}{n}\right).$$

Similar comments apply to functions that satisfy the conditions of Theorem 2 or 3.

Another source of estimates for $E_{n,k}(f)$ are the results of Trigub ([6], p. 263) (see also Malozemov [3]). As a special case, these results contain the following. Let $f^{(r)}$ be continuous on $[-1,+1]$. Then there exists a sequence $Q_n(x)$, $n \geqslant r$, of polynomials of degree n or less, such that for $0 \leqslant s \leqslant r$

$$|f^{(s)}(x) - Q_n^{(s)}(x)| \leqslant C_r n^{s-r}\, \omega\left(f^{(r)}, \frac{1}{n}\right), \tag{14}$$

where C_r is a constant depending only upon r. It follows from this that if $f^{(k)}(x) > 0$ on $[-1,+1]$, $k \leqslant r$, then

$$E_{n,k}(f) \leqslant C_r n^{-r}\, \omega\left(f^{(r)}, \frac{1}{n}\right)$$

for n sufficiently large. Here $E_{n,k}(f)$ is, of course, defined for $[-1,+1]$ in the same way as for $[0,1]$.

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